Math 429 - Exercise Sheet 14

1. Compute the character of the representation $\wedge^i \mathbb{C}^n$ of $\mathfrak{sl}_n$, given our explicit description of its weight spaces in Lecture 13, and use this to verify the Weyl character formula (197).

2. Looking back to Exercise 5 on last week's sheet, compute the character of the tautological representation of $\mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$, respectively (and use this to verify the Weyl character formula).

3. The adjoint representation of any simple Lie algebra is $L(\theta)$, where θ denotes the maximal root (i.e. the unique positive root such that $\theta + \alpha \notin R$ for all $\alpha \in R^+$). Compute the character of the adjoint representation of $\mathfrak{sl}_n$, and verify the Weyl character formula.

4. It is easy to see that $L(0) = \mathbb{C}$ for any semisimple Lie algebra $\mathfrak{g}$ (construct the action explicitly), and so the Weyl character formula implies the equality

$$\sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho)} = \prod_{\alpha \in R^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})$$

Prove this formula directly using the theory of root systems (*Hint: show that both sides of the equation are Weyl group anti-invariant*).

(*) Consider the following inner product of characters

$$(f, g) = \frac{1}{|W|} \int f \overline{g} \prod_{\alpha \in R} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) \tag{1}$$

where $\int e^\lambda = \delta_{\lambda 0}$ and $\overline{e^\lambda} = e^{-\lambda}$. Prove that $(f, g) = (g, f)$ and use the Weyl character formula to show that the characters of irreducible representations are orthogonal with respect to (1).